Combining Several Estimates

j.p.lewis

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Maybeck's Kalman tutorial gives a formula for combining several estimates but does not derive it. Likewise I don't see this explicitly in Papoulis, though the ingredients are all there.

Given two estimates x_1, x_2 of the same quantity x, but assuming that the two estimates are not necessarily equally reliable, and thus assigning differing σ_1, σ_2, how to combine them to form a single estimate?

This can be done by applying the orthogonality principle, and by requiring that the resulting estimate be unbiased. The estimate will be a linear combination of the two observations:

$$\hat{x} = w_1 x_1 + w_2 x_2$$

Starting with the no-bias requirement,

$$E[x - \hat{x}] = 0$$

Rewrite the observations x_1, x_2 as the true value x plus observation noise r_1, r_2.

$$\begin{aligned}
x_1 = x + r_1, &\qquad x_2 = x + r_2 \\
E[x - (w_1(x + r_1) + w_2(x + r_2))] &= 0 \\
x &= w_1(x + E(r_1)) + w_2(x + E(r_2)) \\
&\qquad \text{but if } E(r_1) = E(r_2) = 0 \\
x &= w_1 x + w_2 x \\
w_1 + w_2 = 1 &
\end{aligned}$$

so the weights must sum to one if the observations and the resulting estimate are unbiased. Hmm...

Next apply the orthogonality principle:

$$\begin{aligned}
E[(w_1 x_1 + w_2 x_2 - x) x_1] &= 0 \quad \text{“e1”} \\
E[(w_1 x_1 + w_2 x_2 - x) x_2] &= 0 \quad \text{“e2”}
\end{aligned}$$

$$
\begin{aligned}
\text{expand e1:} & \\
E[w_1(x+r_1)(x+r_1) + w_2(x+r_2)(x+r_1) - x(x+r_1)] &= 0 \\
\text{in} & \\
(x+r_1)(x+r_1) &= x^2 + xr_1 + xr_2 + r_1^2 \\
\text{we assume that } r_1, r_2 \text{ are uncorrelated} & \\
\text{and uncorrelated with } x\text{, so} & \\
\text{e1 after expanding and cancelling uncorrelated terms:} & \\
E[w_1(x^2 + r_1^2) + w_2 x^2 - x] &= 0 \\
\\
\text{do the same with e2:} & \\
E[w_1 x^2 + w_2(x^2 + r_2^2) - x] &= 0 \\
\\
\text{and subtract this from the similarly processed e1} & \\
E[w_1(x^2 + r_1^2 - x^2) + w_2(x^2 - x^2 - r_2^2) &= 0 \\
w_1 E[r_1^2] = w_2 E[r_2^2] & \\
\frac{w_1}{w_2} = \frac{\sigma_2^2}{\sigma_1^2} &
\end{aligned}
$$

Now substitute $w_2 = 1 - w_1$ because the weights sum to 1,

$$
\begin{aligned}
\frac{w_1}{1-w_1} &= \frac{\sigma_2^2}{\sigma_1^2} \\
w_1 &= (1-w_1)\frac{\sigma_2^2}{\sigma_1^2} \\
&= \frac{\sigma_2^2}{\sigma_1^2} - w_1\frac{\sigma_2^2}{\sigma_1^2} \\
w_1\left(1 + \frac{\sigma_2^2}{\sigma_1^2}\right) &= \frac{\sigma_2^2}{\sigma_1^2} \\
w_1 &= \frac{\sigma_2^2}{\sigma_1^2\left(1 + \frac{\sigma_2^2}{\sigma_1^2}\right)} \\
&= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}
\end{aligned}
$$

This is similar to the formula in Maybeck.