Discrete Adjoint and Divergence

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(Assuming regular sampling)

Adjoint

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

In discrete terms

$$\begin{aligned} f'(Dg) &= (f'D)g \\ &= (D'f) \cdot g \end{aligned}$$

adjoint = transpose (in real case)

Case of derivative

Bishop appendix: "adjoint derivative is negative derivative"

$$D = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \cdots & \end{bmatrix}$$

$D' = -D$:

$$D' = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & \cdots & & \end{bmatrix}$$

Relationship to integration by parts

$$uv|_a^b = \int_a^b udv + \int_a^b vdu$$

Discrete version of this is

$$f'Dg = X - g'Df$$

where X is the boundary condition part, often zero.

When X is zero this can be rewritten as

$$f'Dg = -f'D'g$$

Adjoint as cheap inverse

The inverse can be written as

$$(A'A)^{-1}A'$$

Then in some cases the $A'A$ part is nearly identity, or is something simple like a constant scale. Then the inverse is approximately the Adjoint.

Continuous Poisson equation derivation

Seek a function s that minimizes the squared gradient,

$$\min_s I = \int_\Omega \nabla s \cdot \nabla s d\mathbf{x}$$

Boundary condition is gradient vanishes on the boundary, (note my notation, does not follow paper on this point)

$$\frac{\partial s}{\partial \nu} = 0 \quad \text{for } s \in \partial\Omega$$

where ν is a vector perpendicular to the boundary.

Adding a perterbation ϵp,

$$I = \int_\Omega \nabla(s + \epsilon p) \cdot \nabla(s + \epsilon p) d\mathbf{x}$$

M: Expanding the square

$$I = \int_\Omega \|\nabla s\|^2 + 2\nabla s \cdot \nabla(\epsilon p) + \|\nabla(\epsilon p)\|^2 d\mathbf{x}$$

The necessary first order criterion for s to be optimium is

$$\frac{dI}{d\epsilon}|_{\epsilon=0} = 0 = 2\int_\Omega \nabla s \cdot \nabla p d\mathbf{x}$$

Or

$$\int_\Omega \nabla \cdot (p\nabla s) d\mathbf{x} \quad - \quad \int_\Omega [p\nabla^2 s] d\mathbf{x} = 0$$

Using the divergence theorem on the first term,

$$\int_\Omega \nabla \cdot (p\nabla s) d\mathbf{x} = \int_\Omega p\nabla s \cdot \nu dS$$

and this vanished because of the boundary condition, leaving

$$\int_\Omega [p\nabla^2 s] d\mathbf{x} = 0$$

Discrete Poisson equation derivation

$$\min_f \|Gf\|^2 = \min_f f'G'Gf \quad \Rightarrow \quad G'Gf = 0$$