Relaxation Notes

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General strategy: elliptic eq 'static' form (my term) such as $\nabla^2 f = 0$ also has a corresponding formulation which minimizes an energy, $\int (\nabla f)^2 dxy$. This can be shown by starting with the energy minimization form and adding a variational calculus 'variation', integrate by parts, get the 'static' form.

Easier, see below, is to discretize, then minimize wrt f_k, you get the static form as a result.

$$\min_f \int (\nabla f)^2 dxy \qquad = \min_f \sum (\nabla f)^2$$

$$\nabla f = (\frac{df}{dx}, \frac{df}{dy})' \qquad \approx \frac{1}{h}(f_{10} - f_{00}), \frac{1}{h}(f_{01} - f_{00}) \text{ using forward first diff}$$

$$(\nabla f)^2 \qquad = \frac{1}{h^2}((f_{10} - f_{00})^2 + (f_{01} - f_{00})^2)$$

$$E = \sum (\nabla f)^2 \qquad = \frac{1}{h^2} \sum \ldots + (f_{k,j} - f_{k-1,j})^2 + (f_{k,j} - f_{k,j-1})^2 + (f_{k+1,j} - f_{k,j})^2 + (f_{k,j+1} - f_{k,j}$$

$$\frac{dE}{df_{k,j}} \qquad = \frac{1}{h^2}\left[(f_{k,j} - f_{k-1,j}) + (f_{k,j} - f_{k,j-1}) - (f_{k+1,j} - f_{k,j}) - (f_{k,j+1} - f_{k,j})\right]$$

$$= \frac{1}{h^2}(4 f_{k,j} - f_{k+1,j} - f_{k-1,j} - f_{k,j+1} - f_{k,j-1})$$

now iterate, reduce the error by changing f: $\qquad f_{k,j} \leftarrow f_{k,j} - \alpha \frac{dE}{df_{k,j}}$

α needs to be 1/4 or less for stability, see NR

$$f_{k,j} \leftarrow f_{k,j} - 1/4(4 f_{k,j} - f_{k+1,j} - f_{k-1,j} - f_{k,j+1} - f_{k,j-1})$$

$$f_{k,j} \leftarrow 1/4(f_{k+1,j} + f_{k-1,j} + f_{k,j+1} + f_{k,j-1})$$

Now compare this to the solution starting from the static form:

forward first difference $\qquad \frac{1}{h}(f_+ - f_0)$

2nd difference = difference of first diff $\qquad \frac{1}{h}(\frac{1}{h}(f_+ - f_0) - \frac{1}{h}(f_0 - f_-))$

$$= \frac{1}{h^2}(f_+ - 2f_0 + f_-)$$

so $\qquad \nabla^2 f = 0 = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{1}{h^2}\left[f_{+0} - 2f_{00} + f_{-0}\right] + \frac{1}{h^2}\left[f_{0+} - 2f_{00} + f_{0-}\right]$

$$= \frac{1}{h^2}(f_{+0} + f_{-0} + f_{0+} + f_{0-} - 4f_{00}) = 0$$

since this equals zero, can mult by h^2 $\qquad = f_{+0} + f_{-0} + f_{0+} + f_{0-} - 4f_{00} = 0$

Now write this as a linear system and show its solution with Jacobi iteration. It can be written as $Af = 0$ with f unrolled into a vector and A a len(f)^2 matrix. Rows of this matrix look like

$$\ldots 1, \ldots * \ldots, 1, -4, 1, \ldots * \ldots, 1, \ldots$$

where * is the offset from one row to the previous or next, and -4 is on the diagonal.

Jacobi iteration sets each f_k to the solution of its row of the matrix equation, independent of all other rows:

$$\sum A_{rc} f_c = b_r$$

$$\rightarrow \qquad A_{rk} f_k = b_k - \sum_{j \neq k} A_{rj} f_j$$

$$f_k \leftarrow \frac{b_k}{A_{kk}} - \sum_{j \neq k} A_{kj}/A_{kk} f_j$$

Apply this here:

$$\begin{aligned}
\text{eqn of row k:} \quad & -4f_k + \text{ surrounding points } = 0 \\
& f_k \leftarrow 1/4 \text{ surrounding points} \\
\text{converting back to 2d, gives same result as energy min} \quad & f_{k,j} \leftarrow 1/4(f_{k+1,j} + f_{k-1,j} + f_{k,j+1} + f_{k,j-1})
\end{aligned}$$

The Jacobi iteration converges if the matrix is 'diagonally dominant', meaning that the diagonal elements are much larger than the off diagonal elements. Strict row-diagonal dominance requires that the abs of the diagonal be greater than the sum-abs of the off-diagonal elements.

For *interpolation* some $f_{r,c}$ are specified rather than free. In setting up the linear system, subtract these from both sides of the eq, so the known quantities move to the rhs. CHECK:

$$\begin{aligned}
\frac{1}{h^2}\left(f_{+0} + f_{-0} + f_{0+} + f_{0-} - 4f_{00}\right) = 0 & \qquad \text{say } f_{+0} \text{ is known/fixed, then} \\
\frac{1}{h^2}\left(f_{-0} + f_{0+} + f_{0-} - 4f_{00}\right) = -\frac{1}{h^2} f_{+0} &
\end{aligned}$$

TODO: show interp derivative

can square of gradient be expressed as a matrix?

$$\begin{aligned}
|\nabla u|^2 = (\frac{du}{dx})^2 + (\frac{du}{dy})^2 & \\
\text{say } K_x \text{ is the matrix that produces } du/dx\text{, then} \quad & u' K_x' K_x u = (du/dx)^2 \\
\text{so } u'(K_x' K_x + K_y' K_y)u \quad & = (u' K_x' K_x + u' K_y' K_y)u \\
= u' K_x' K_x u + u' K_y' K_y u \quad & = (\frac{du}{dx})^2 + (\frac{du}{dy})^2 \text{ try this in maple}
\end{aligned}$$

minimizing gradient squared same as laplace

Dan Ruderman derives this, requires several integration by parts. A simpler approach is (in 1d):

$$\begin{aligned}
\text{roughness} \quad & R = \int |\nabla u|^2 du \approx \sum (u_{k+1} - u_k)^2 \\
\text{for a particular k:} \quad & \frac{dR}{du_k} = \frac{d}{du_k}[(u_k - u_{k-1})^2 + (u_{k+1} - u_k)^2] \\
& = 2(u_k - u_{k-1}) - 2(u_{k+1} - u_k) = 0 \\
& u_{k+1} - 2u_k + u_{k-1} = 0 \rightarrow \nabla^2 u = 0
\end{aligned}$$