General strategy: elliptic eq 'static' form (my term) such as $\nabla^2 f = 0$ also has a corresponding formulation which minimizes an energy, $\int (\nabla f)^2 \, dxy$. This can be shown by starting with the energy minimization form and adding a variational calculus 'variation', integrate by parts, get the 'static' form.

Easier, see below, is to discretize, then minimize wrt $f_k$, you get the static form as a result.

$$\min_f \int (\nabla f)^2 \, dxy = \min_f \sum (\nabla f)^2$$

$$\nabla f = \left( \frac{df}{dx}, \frac{df}{dy} \right)' \approx \frac{1}{h} (f_{o+} - f_{o0}), \frac{1}{h} (f_{0+} - f_{00}) \text{ using forward first diff}$$

$$(\nabla f)^2 = \frac{1}{h^2} ((f_{o+} - f_{o0})^2 + (f_{0+} - f_{00})^2)$$

$$E = \sum (\nabla f)^2 = \frac{1}{h^2} \sum \ldots + (f_{k,j} - f_{k-1,j})^2 + (f_{k,j} - f_{k,j-1})^2 + (f_{k+1,j} - f_{k,j})^2 + (f_{k,j+1} - f_{k,j})^2$$

$$\frac{dE}{df_{k,j}} = \frac{1}{h^2} [ (f_{k,j} - f_{k-1,j}) + (f_{k,j} - f_{k,j-1}) - (f_{k+1,j} - f_{k,j}) - (f_{k,j+1} - f_{k,j})]$$

$$= \frac{1}{h^2} (4f_{k,j} - f_{k-1,j} - f_{k-1,j} - f_{k,j+1} - f_{k,j-1})$$

now iterate, reduce the error by changing f:

$$f_{k,j} \leftarrow f_{k,j} - \alpha \frac{dE}{df_{k,j}}$$

\(\alpha\) needs to be 1/4 or less for stability, see NR

$$f_{k,j} \leftarrow f_{k,j} - 1/4(4f_{k,j} - f_{k+1,j} - f_{k-1,j} - f_{k,j+1} - f_{k,j-1})$$

$$f_{k,j} \leftarrow f_{k,j} - 1/4(f_{k+1,j} + f_{k-1,j} + f_{k,j+1} + f_{k,j-1})$$

Now compare this to the solution starting from the static form:

forward first difference

$$\frac{1}{h} (f_{o+} - f_o)$$

2nd difference = difference of first diff

$$\frac{1}{h} \left( \frac{1}{h} (f_{o+} - f_0) - \frac{1}{h} (f_0 - f_-) \right)$$

$$= \frac{1}{h^2} (f_+ - 2f_0 + f_-)$$

so

$$\nabla^2 f = 0 = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{1}{h^2} [f_{o+} - 2f_{00} + f_{o-}] + \frac{1}{h^2} [f_{0+} - 2f_{00} + f_{0-}]$$

$$= \frac{1}{h^2} (f_{o+} + f_{o-} - f_{0+} + f_{0-} - 4f_{00}) = 0$$

since this equals zero, can mult by $h^2$

$$= f_{o+} + f_{o-} + f_{0+} + f_{0-} - 4f_{00} = 0$$

Now write this as a linear system and show its solution with Jacobi iteration. It can be written as $Af = 0$ with f unrolled into a vector and A a len(f)^2 matrix. Rows of this matrix look like

$$\ldots 1, \ldots * \ldots 1, -4, 1, \ldots * \ldots 1, \ldots$$

where * is the offset from one row to the previous or next, and -4 is on the diagonal.

Jacobi iteration sets each $f_k$ to the solution of its row of the matrix equation, independent of all other rows:

$$\sum A_{re} f_e = b_r$$

$$\rightarrow A_{rk} f_k = b_k - \sum_{j \neq k} A_{rj} f_j$$

...
\[ f_k \leftarrow \frac{b_k}{A_{kk}} - \sum_{j \neq k} A_{kj}/A_{kk} f_j \]

Apply this here:

eqn of row k:
\[-4f_k + \text{surrounding points} = 0\]
\[ f_k \leftarrow 1/4( f_{k+1,j} + f_{k-1,j} + f_{k,j+1} + f_{k,j-1} ) \]

The Jacobi iteration converges if the matrix is ‘diagonally dominant’, meaning that the diagonal elements are much larger than the off diagonal elements. Strict row-diagonal dominance requires that the abs of the diagonal be greater than the sum-abs of the off-diagonal elements.

For interpolation some \( f_{r,c} \) are specified rather than free. In setting up the linear system, subtract these from both sides of the eq, so the known quantities move to the rhs. CHECK:

\[ \frac{1}{h^2} (f_{+0} + f_{-0} + f_{0+} + f_{0-} - 4f_{00}) = 0 \quad \text{say } f_{+0} \text{ is known/fixed, then} \]
\[ \frac{1}{h^2} (f_{-0} + f_{0+} + f_{0-} - 4f_{00}) = -\frac{1}{h^2} f_{+0} \]

TODO: show interp derivative

**can square of gradient be expressed as a matrix?**

\[ |\nabla u|^2 = (\frac{du}{dx})^2 + (\frac{du}{dy})^2 \]

say \( K_x \) is the matrix that produces \( du/dx \), then

\[ u' K' x K x u = (du/dx)^2 \]

so \( u' (K'_x K_x + K'_y K_y) u \)

\[ = (u' K'_x K_x + u' K'_y K_y) u \]

\[ = \frac{du}{dx}^2 + \frac{du}{dy}^2 \] try this in maple

**minimizing gradient squared same as laplace**

Dan Ruderman derives this, requires several integration by parts. A simpler approach is (in 1d):

\[ R = \int |\nabla u|^2 du \approx \sum (u_{k+1} - u_k)^2 \]

for a particular k:

\[ \frac{dR}{du_k} = \frac{d}{du_k} \left[ (u_k - u_{k-1})^2 + (u_{k+1} - u_k)^2 \right] \]

\[ = 2(u_k - u_{k-1}) - 2(u_{k+1} - u_k) = 0 \]

\[ u_{k+1} - 2u_k + u_{k-1} = 0 \rightarrow \nabla^2 u = 0 \]