Several Simple Applications of Eigenvectors in Graphics

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version 0.xx may 2004
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Some of the CG textbooks stop short of using eigenvalues/vectors, but they are needed for understanding current research in graphics. This pdf gives several simple examples of how eigenvectors are useful in graphics problems.

Line fitting

For 1D lines we often use the one-dimensional y=f(x) parameterization. Fitting a line using this preresentation is easy, but it only works well when the line slope is shallow. When the slope is steep the error used in the fit is not between the data point and the closest point on the line (as would be desired), rather it is between the data point and and the point on the line that is vertically above or below it.

In *total least squares* the fit is measured as the sum squared distance between the data and their closest points on the line. This approach generalizes to fitting hyperplanes.

The standard line representation is $y = ax + c$, rewrite this as $1 \cdot y - a \cdot x = c$, or $(1, -a)^T(y, x) = c$; call this

$$\mathbf{a}^T\mathbf{x} = c$$

(a hyperplane).

Now minimize the squared distances to the line

$$\min \sum (\mathbf{a}^t\mathbf{x}_k - c)^2$$

subject to $\|\mathbf{a}\| = 1$.

$\mathbf{a}^T\mathbf{x} - c$ is the distance to the line ($\mathbf{a}^T\mathbf{x} + c$ might look more familiar, it would be the distance if we had used the hyperplane equation $\mathbf{a}^T\mathbf{x} + c = 0$.) Note that $\mathbf{a}$ and c can be scaled without changing the plane, so scale so that the normal vector $\mathbf{a}$ has length 1 to eliminate this freedom.

$$\begin{aligned}
\min_{a,c} \quad & \sum(\mathbf{a}'\mathbf{x}_k - c)^2 + \lambda(\mathbf{a}'\mathbf{a} - 1) \\
& = \sum a'x_kx_k'a - 2\sum ca'x_k + \sum c^2 + \lambda(a'a - 1) \\
& = a'(\sum x_kx_k')a - 2ca'\sum x_k + \sum c^2 + \lambda(a'a - 1) \\
\text{call} \quad & X \equiv \sum x_kx_k' \text{ and } \hat{x} \equiv \sum x_k \\
& = a'Xa - 2ca'\hat{x} + Nc^2 + \lambda(a'a - 1) \\
\frac{d}{da} = 0 \quad & = 2Xa - 2c\hat{x} + 2\lambda a \\
\frac{d}{dc} = 0 \quad & = 2a'\hat{x} + 2Nc = 0 \rightarrow c = a\hat{x}/N \\
\text{substitute c} \quad & 2Xa - 2(a'\hat{x})\hat{x} + 2\lambda a = 0 \\
& = Xa - \hat{x}(a'\hat{x}) + \lambda a = 0 \\
& = Xa - \hat{x}(\hat{x}'a) + \lambda a = 0 \\
& = (X - \hat{x}\hat{x}')a + \lambda a = 0
\end{aligned}$$

which is an eigenvalue problem, and the minimum eigenvalue minimizes the original problem, with the corresponding eigenvector being the desired coefficient (normal) vector a.

Find the axis of rotation given the matrix

Given a rotation matrix R, if x is the rotation axis then $Rx = x$. So $Rx - x = 0$, or $(R - I)x = 0$. The null space of $R - I$ is therefore the axis of rotation.

Take the svd,

$$\begin{aligned} & Rx = (UDV)x = 0 \\ \text{U does not affect length of x so it can be ignored} \quad & (DV)x = 0 \\ \text{call } Vx = y\text{:} \quad & Dy = 0 \\ \text{y can only be nonzero where the eigenvals are zero} \quad & \\ \text{last cols of V span the nullspace} \quad & x = V'y \end{aligned}$$

So the last column of V is the rotation axis.

Columns of V are the eigenvectors of the matrix $R^T R$.